The non-existence of a Lanczos potential for the Weyl curvature tensor in dimensions $n \geq 7$

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Abstract

In this paper it is shown that a Lanczos potential for the *Weyl curvature tensor* does not exist for all spaces of dimension $n \geq 7$.

KEYWORDS: Lanczos potential; existence; Weyl curvature tensor

Whether there exists a Lanczos potential [1] for Weyl curvature tensors in dimensions $n > 4$ has still not been determined. Although Lanczos's original proof [1] for existence was flawed [2], there have subsequently been complete proofs for existence in *four* dimensions [2, 3, 4]. The latter two proofs [3, 4] are in spinors, although in [4] a translation into tensors is given which is explicitly for *four* dimensions, independent of signature. The detailed and complicated proof given by Bampi and Caviglia [2] is also explicitly for *four* dimensions, although they also discuss briefly the possibility of existence in higher dimensions.

An important aspect of all of these proofs is that they are not only valid for Weyl curvature tensors C_{abcd}, but for *Weyl candidate tensors* W_{abcd}, i.e., any 4-tensor having the index symmetries of the Weyl curvature tensor,

$$W_{abcd} = W_{[ab]cd} = W_{ab[cd]} = W_{cdab} \quad W^a{}_{bad} = 0. \tag{1a}$$

$$0 = W_{a[bcd]} \tag{1b}$$

In a recent paper [5] we have shown that a Lanczos potential for a Weyl candidate tensor does not *generally* exist for dimensions $n > 4$. In particular, we have shown that in flat and conformally flat spaces with dimensions $n > 4$, the assumed existence of a Lanczos potential for an arbitrary Weyl candidate imposes non-trivial conditions on the Weyl candidate. However, this result does not say anything about the existence of a Lanczos potential for a *Weyl curvature tensor* in spaces with non-zero conformal curvature in dimensions $n > 4$.

In this paper we will address the problem of Weyl curvature tensors *directly*, and show explicitly, that

A Lanczos potential for the Weyl curvature tensor does not exist for all spaces of dimension $n \geq 7$.

The n-dimensional generalisation for the Lanczos potential of a Weyl candidate is given by [6]

$$W^{ab}{}_{cd} = 2L^{ab}{}_{[c;d]} + 2L_{cd}{}^{[a;b]} - \frac{4}{(n-2)}\delta^{[a}_{[c}\left(L^{b]i}{}_{d];i} + L_{d]i}{}^{b];i}\right) \tag{2}$$

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where the Lanczos tensor L_{abc} has the properties that

$$L_{abc} = L_{bac} \tag{3a}$$

$$L_{ab}{}^{b} = 0 \tag{3b}$$

$$L_{[abc]} = 0. \tag{3c}$$

It is easy to check that W_{abcd} satisfies the defining equations (1) for the Weyl candidate tensor.

The condition (3b) is called the Lanczos algebraic gauge and can always be assumed without loss of generality. However, equation (2) has a more complicated appearance when the Lanczos algebraic gauge is not imposed.

By checking integrability conditions, it was shown in [5] that — *in dimen-*

sions $n \geq 6$ — (2) leads to the long complicated condition

$$
\begin{aligned}
W^{[ab}{}_{[cd;e]}{}^{f]} = {} & L_{[cd}{}^{[a;|i|}C^{bf]}{}_{e]i} - L_{[cd}{}^{i;[a}C^{bf]}{}_{e]i} - 2L_{[c}{}^{i[a}{}_{;d}C^{bf]}{}_{e]i} - 2L_{[c}{}^{i[a;b}C^{f]}{}_{|i|de]} \\
& - 2L^{[a|i|}{}_{[c}{}^{;b}C^{f]}{}_{|i|de]} + L_{[c}{}^{i[a}C^{bf]}{}_{de];i} + L^{[a|i|}{}_{[c}C^{bf]}{}_{de];i} \\
& - \frac{2}{n-4}\delta^{[a}_{[c}\Big(L_{de]}{}^{|i;j|}C^{bf]}{}_{ij} + 2L_{d}{}^{|i|b;|j|}C^{f]}{}_{|j|e]i} + 2L^{b|i|}{}_{d}{}^{;|j|}C^{f]}{}_{|i|e]j} \\
& \qquad + L_{d}{}^{|ij|}{}_{;e]}C^{bf]}{}_{ij} + L^{b|ij|;f]}C_{de]ij} + L_{d}{}^{|ij|}{}_{;j}C^{bf]}{}_{e]i} + L^{b|ij|}{}_{;|j|}C^{f]}{}_{|i|de]} \\
& \qquad + L_{d}{}^{|ij|}{}_{;i}C^{bf]}{}_{e]j} + L^{|ij|}{}_{d}{}^{;b}C^{f]}{}_{e]ij} - L^{|ij|b}{}_{;d}C^{f]}{}_{e]ij} + L^{|ij|b}{}_{;|i|}C^{f]}{}_{|j|de]} \\
& \qquad - L^{|ij|b;f]}C_{de]ij} + L_{d}{}^{|ij|}C^{bf]}{}_{e]j;i} + L^{b|ij|}C_{de]}{}^{f]}{}_{j;i} - \frac{1}{2}L^{|ij|}{}_{d}C^{bf]}{}_{|ij|;e]} \\
& \qquad - \frac{1}{2}L^{|ij|b}C_{de]ij}{}^{;f]} + \frac{2}{n-3}L_{d}{}^{|i|b}C^{f]}{}_{|i|e]j;}{}^{j} + \frac{2}{n-3}L^{b|i|}{}_{d}C_{e]i}{}^{f]}{}_{j;}{}^{j}\Big) \\
& - \frac{2}{(n-3)(n-4)}\delta^{[a}_{[c}\delta^{b}_{d}\Big(2L_{e]}{}^{|ij;k|}C^{f]}{}_{jki} + 2L^{f]ij;k}C_{|ijk|e]} - L^{|ij|}{}_{e]}{}^{;|k|}C_{ijk}{}^{f]} \\
& \qquad - 2L^{|ij|f];k}C_{|ijk|e]} + L^{|ijk|}{}_{;e]}C_{ijk}{}^{f]} + 2L^{|ijk|;f]}C_{|ijk|e]} + L^{|ijk|}C_{|ijk|e]}{}^{;f]} \\
& \qquad + L^{|ijk|}C_{|ijk|}{}^{f]}{}_{;e]} + \frac{2}{n-3}L_{e]}{}^{|ij|}C^{f]}{}_{ijk;}{}^{k} + \frac{2}{n-3}L^{f]ij}C_{e]ijk;}{}^{k} \\
& \qquad + \frac{n-2}{n-3}L^{|ij|}{}_{e]}C_{ij}{}^{f]}{}_{k;}{}^{k} + \frac{n-2}{n-3}L^{|ij|f]}C_{|ij|e]k;}{}^{k}\Big) \\
& + \frac{4}{(n-3)^2(n-4)}\delta^{[a}_{[c}\delta^{b}_{d}\delta^{f]}_{e]}L^{ijk}C_{ijkl;}{}^{l} \\
& + \frac{1}{(n-2)(n-3)(n-4)}\delta^{[a}_{[c}\delta^{b}_{d}\delta^{f]}_{e]}C_{ijkl}W^{ijkl} \\
& + \frac{1}{n-4}\delta^{[a}_{[c}W^{bf]}{}_{de];i}{}^{i} + \frac{2}{n-4}\delta^{[a}_{[c}W^{|i|b}{}_{de];i}{}^{f]} + \frac{2}{n-4}\delta^{[a}_{[c}W^{bf]}{}_{|i|d;e]}{}^{i} \\
& + \frac{4}{(n-3)(n-4)}\delta^{[a}_{[c}\delta^{b}_{d}W^{f]i}{}_{e]j;i}{}^{j} - \frac{2}{n-2}W^{[a}{}_{[cd}{}^{b}\tilde{R}^{f]}{}_{e]} \\
& - \frac{4}{(n-2)(n-4)}\delta^{[a}_{[c}W^{b|i|f]}{}_{d}\tilde{R}_{e]i} - \frac{4}{(n-2)(n-4)}\delta^{[a}_{[c}W^{b}{}_{de]}{}^{|i|}\tilde{R}^{f]}{}_{i} \\
& - \frac{4}{(n-2)(n-3)(n-4)}\delta^{[a}_{[c}\delta^{b}_{d}W^{f]i}{}_{e]}{}^{j}\tilde{R}_{ij}
\end{aligned}
\tag{4}
$$

where $\tilde{R}_{ab}$ is the trace free Ricci tensor. In conformally flat spaces (i.e., $C_{abcd} = 0$) all the terms explicitly containing L_{abc} disappear, and we obtain, in general, a nontrivial effective restriction on the Weyl candidate W_{abcd}. However, when we specialise the Weyl candidate W_{abcd} to the Weyl curvature tensor C_{abcd} then this restriction collapses in conformally flat spaces.

Let us now consider a particular subclass of $n \geq 7$ dimensional spaces (where we now use coordinate index notation and let lower case Latin letters range from 1 to n, lower case Greek letters from 1 to 4 and capital Greek letters from 5 to

n) with the metric

$$
\begin{aligned}
ds^2 &= g_{ab}dx^a dx^b \\
&= g_{\alpha\beta}dx^\alpha dx^\beta + (dx^5)^2 + (dx^6)^2 + (dx^7)^2 + \ldots + (dx^n)^2 \\
&= g_{\alpha\beta}dx^\alpha dx^\beta + \eta_{\Sigma\Omega}dx^\Sigma dx^\Omega
\end{aligned} \tag{5}
$$

where $g_{\alpha\beta}$ is a *Ricci flat* 4-dimensional metric, i.e., $g_{\alpha\beta}$ depends only on x^1, ..., x^4 and its four dimensional Ricci tensor is zero.

The following properties follow,

- the n-dimensional space is Ricci flat, i.e., $R_{ab} = 0$, and so $R_{abcd} = C_{abcd}$ and from the Bianchi identities $C_{abc}{}^d{}_{;d} = 0$.
- all Weyl tensor coordinate components with at least one entry of $5, 6, 7, ..., n$ are zero, i.e.,

$$C_{\Sigma bcd} = 0 = C^\Sigma{}_{bcd}, \tag{6}$$

  together with all other components from symmetry properties and index raising by metric.
- all Christoffel symbols with at least one entry of $5, 6, 7, ...n$ are zero, i.e.,

$$\Gamma^a_{b\Sigma} = \Gamma^\Sigma_{ab} = 0 \tag{7}$$

  together with all other components from symmetry and metric properties.
- all derivatives of Weyl tensor whose coordinate components have at least one entry of $5, 6, 7, \ldots, n$ are zero, i.e.,

$$C_{\Sigma bcd;e} = C_{\Sigma bcd;ef} = \cdots = 0 = C^\Sigma{}_{bcd;e}\cdots, \tag{8}$$

  together with all other components from symmetry properties.

When we use this n-dimensional metric with the substitutions $(a, b, f) = (5, 6, 7) = (c, d, e)$ and $W_{ijkl} = C_{ijkl}$ the constraint (4) simplifies a lot because almost all of the products contain a term with $C_{\Sigma bcd}$. So (4) becomes

$$C_{ijkl}C^{ijkl} = 0. \tag{9}$$

This is a restriction on the metric (5), which translates directly to the 4-dimensional Ricci-flat metric $g_{\alpha\beta}$; since it requires a zero value for a normally non-zero Riemann scalar invariant of this metric, (even in Ricci flat case, e.g., Schwarzschild) it is therefore a non-trivial effective restriction.

If we do the same, but without setting $W_{abcd} = C_{abcd}$, we get a restriction on the *Weyl candidate* tensor,

$$
\begin{aligned}
W^{[ab}{}_{[cd;e]}{}^{f]} = {} & \frac{1}{(n-2)(n-3)(n-4)}\delta^{[a}_{[c}\delta^b_d\delta^{f]}_{e]}C_{ijkl}W^{ijkl} \\
& + \frac{1}{n-4}\delta^{[a}_{[c}W^{bf]}{}_{de];i}{}^i + \frac{2}{n-4}\delta^{[a}_{[c}W^{|i|b}{}_{de];i}{}^{f]} \\
& + \frac{2}{n-4}\delta^{[a}_{[c}W^{bf]}{}_{|i|d;e]}{}^i + \frac{4}{(n-3)(n-4)}\delta^{[a}_{[c}\delta^b_d W^{f]i}{}_{e]j;i}{}^j
\end{aligned} \tag{10}
$$

This is an effective restriction since $W_{abcd} = C_{abcd}$ does not satisfy it. It is more general than it first appears to be; in the derivation the conditions (1b) and (3c) have not been used. Condition (10) is therefore also a restriction on the existence of a Lanczos potential for the larger class of Weyl candidate tensors which lack the symmetry (1b). The existence problem without the conditions (1b) and (3c) is referred to as the *parallel problem* in [5] and [2].

Finally we emphasise that although condition (4) is also applicable to spaces of dimension six, when we construct an analogous metric to (5) it does not have the same crucial properties in 6-dimensional spaces, and so we cannot draw the same conclusions as above. As regards 5-dimensional spaces, we showed in [5] that condition (4) is trivially satisfied, but that another — even more complicated — condition applies (involving third derivatives of the Lanczos potential, and too complicated to write out explicitly). So the question of existence of a Lanczos potential specifically for the Weyl curvature tensor in all spaces of dimensions five and six has still not been formally ruled out.

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